Was an Industrial Revolution Inevitable?
Economic Growth Over the Very Long Run

Charles I. Jones
Stanford University and NBER
Chad.Jones@Stanford.edu
http://www.stanford.edu/~chadj
May 26, 1999 – Version 1.5

Abstract
This paper studies a growth model that is able to match two key facts of economic history. First, for thousands of years, the average standard of leaving seems to have risen very little, despite increases in the level of technology and large increases in the level of the population. Second, after thousands of years of little change, the level of per capita consumption has increased dramatically in less than two centuries. Quantitative analysis of the model highlights the importance of increases in the productivity with which a given population produces new ideas as crucial to the observed transition to modern economic growth.

JEL Classification: O40, E10

I would like to thank Oded Galor, Avner Greif, Lant Pritchett, Antonio Rangel, Paul Romer, and seminar participants at the Minneapolis Fed, NYU, and Penn State for their comments and suggestions. Financial support from the Stanford Institute for Economic Policy Research and the National Science Foundation (SBR-9510916, SBR-9818911) is gratefully acknowledged.
1 Introduction

The past century has been marked by extremely rapid increases in standards of living. Measured GDP per capita is perhaps ten times higher in the United States today than 125 years earlier, and with a mismeasurement of growth of one percent per year, the factor could easily be more than thirty.

Also remarkable is the relatively short span of world history during which this rapid growth has occurred. Conservative estimates suggest that humans were already distinguishable from other primates 1 million years ago. Imagine placing a time line corresponding to this million year period along the length of a football field. On this time line, humans were hunters and gatherers until the agricultural revolution, perhaps 10,000 years ago — that is, for the first 99 yards of the field. The height of the Roman empire occurs only 7 inches from the rightmost goal line, and the Industrial Revolution begins less than one inch from the field’s end. Large increases in standards of living have occurred during a relatively short time — equivalent to the width of a golf ball resting at the end of a football field.

This paper combines an idea-based theory of growth in which people are a key input into the production of new ideas with a model of endogenous fertility and mortality in order to analyze these remarkable facts. The internal dynamics provided by the model are able to produce thousands of years of virtually no sustained growth in standards of living despite increases in both technology and population, followed by the emergence of rapid growth during an “industrial revolution.” More generally the model matches the broad time series behavior of both population and per capita consumption.

To match the population data exactly, however, the quantitative analysis introduces two shocks. The first shock is a temporary decline in the standard Solow (1956) measure of total factor productivity, as might occur during times of war or famine. The second shock also corresponds to a Solow (1956) residual, but in this case for the production function for new ideas. A
specific pattern of shocks raising the productivity with which the population produces ideas is needed to match the time path of world population. By combining a single model for analyzing both the pre-industrial revolution era and the modern era with the time series data for world population, we are able to identify the “shape” of the production function for new ideas.

Quantitative analysis of the model produces three main findings. First, there is a general upward trend in the productivity of the idea production function. Although this finding may seem trivial, it is important to recognize that from a theoretical standpoint other possibilities are quite plausible. For example, perhaps the most obvious ideas are discovered first, in which case it might actually become harder and harder to discover new ideas over time. By the 20th century, the calibration suggests that the annual rate of production of new ideas, measured in units of output, is about 15,000 times larger than it was in the year 25000 B.C. A factor of 100 of this increase is due to the fact that the world has a larger population available to produce ideas; the remaining factor of 150 reflects the fact that the population is more productive at producing ideas.

Second, consider the following simple production function for new ideas:

$$\Delta A_{t+1} = \delta N_t^\lambda A_t^\phi$$

where $N$ represents research labor and $A$ represents the stock of ideas, so that the change in $A$ over time is the number of new ideas produced. Several papers in the growth literature are concerned with the magnitude of $\phi$. Romer (1990), Grossman and Helpman (1991), and Aghion and Howitt (1992) have $\phi = 1$. Jones (1995), Kortum (1997), and Segerstrom (1998) have $\phi < 1$. In principle, the observations on the shape of the idea production function correspond to $A^\phi$, so that one can obtain a direct measure of $\phi$. The second quantitative finding is that it is difficult to reconcile the observations on the productivity of the idea production function with a stable $A^\phi$ function. The observations suggest a very gradual increase for most of history, followed by an extremely sharp increase in recent centuries.
An alternative interpretation of the evidence is that shifts in this production function — changes in $\delta$ — are critically important. Viewed literally, such shifts simply say that for unmodeled reasons, the world population is better at producing ideas in recent centuries. Viewed from a broader perspective, the shocks seem to indicate that changes in property rights and institutions may be very important. Changes such as the strengthening of property rights, the development of the modern patent system, the establishment of research universities, and the growing linkages between science and industry likely had two important effects. First, they led people to search actively for ideas: a given population is likely to exert much greater effort when the rewards from invention can be internalized. Second, they increased the diffusion of knowledge, making a given amount of effort more productive at generating new ideas.

This project builds on a number of recent papers that study growth over the very long run, including Lee (1988), Becker, Murphy and Tamura (1990), Kremer (1993), Goodfriend and McDermott (1995), Tamura (1998), Lucas (1998), Galor and Weil (1998), and Hansen and Prescott (1998). Like Lee (1988), and Kremer (1993) (and Romer (1990), Simon (1986), and others), the link between population and the discovery of new ideas plays a critical role. Like the human capital-driven models of Becker et al. (1990), Tamura (1998), Lucas (1998), and Galor and Weil (1998), fertility behavior is governed by utility maximization. Common to all of these papers and to this one is a Malthusian building block: a fixed supply of land that generates decreasing returns to scale when technology is held constant.

This paper differs from the existing literature primarily in its emphasis on quantitative theory, i.e. in providing a complete quantitative analysis of the growth model. In addition, however, a few modeling differences are worth noting.

The existing papers on this topic that have included a model of the demographic transition — Becker et al. (1990), Tamura (1998), Lucas (1998),
and Galor and Weil (1998) — do so purely through a fertility transition that occurs as individuals begin to trade off quantity for quality.\textsuperscript{1} The models are set up in an overlapping generations context so that mortality is unaffected by technological progress. In fact, in the traditional formulation of the demographic transition, summarized by Cohen (1995) and Easterlin (1996), the initial rise in population growth is driven primarily by a decline in mortality rather than by a rise in fertility. This paper explicitly models mortality and its dependence on technological progress, and a decline in mortality plays a crucial role in the demographic transition. The subsequent decline in fertility that completes the demographic transition occurs through a substitution effect associated with the way preferences are modeled.

In contrast to many endogenous growth models which emphasize constant returns to accumulable factors, this model emphasizes the importance of increasing returns to factors that can be accumulated (including labor). The nonrivalry of ideas is an important feature in generating increasing returns, but it is not sufficient, due to the presence of land as a fixed factor. For the model to generate the accelerating rate of population growth emphasized by Kremer (1993), and for it to generate an industrial revolution endogenously, the nonrivalry of ideas must be sufficiently strong so as to overcome the diminishing returns implied by the fixed supply of land.

Was an industrial revolution inevitable? The different papers that have looked into this question reach, sometimes implicitly and sometimes explicitly, different conclusions. On the one hand, in the models in most of the papers mentioned above, especially Galor and Weil (1998) and Hansen and Prescott (1998), the dynamics in place from the beginning of time suggest that something like an industrial revolution is inevitable. On the other hand, the model in Lucas (1998) explicitly requires an exogenous shock to the rate of return to human capital accumulation in order to get the indus-

\textsuperscript{1}Galor and Weil (1996) generate a demographic transition through a difference in the endowments of men and women and a shift in comparative advantage. See Galor and Weil (1999) for an overview of several different theories of the demographic transition.
trial revolution going. From a theoretical standpoint, one might imagine that this is an undesirable outcome, but from a historical standpoint — which might emphasize the development of property rights and the advent of science-based research — such a finding may be entirely appropriate.

The present paper is somewhere in between. Something like an industrial revolution is inevitable in the model, at least for a range of parameter values. However, the timing of this industrial revolution is quite sensitive to the parameter values and the nature of the shocks. A counterfactual experiment at the end of the paper suggests, for example, that absent the large productivity shocks to the production function for new ideas that have occurred since the 18th century, the Industrial Revolution would have occurred 400 years later than it actually did.

The remainder of the paper is organized as follows. Section 2 presents the basic model. Section 3 analyzes the model’s dynamics and discusses how it generates a demographic transition. Section 4 presents a summary of the facts the model should address, explains how parameter values are obtained, and exhibits the basic simulation of the model. Section 5 conducts the quantitative analysis. Section 6 discusses some of the results and implications, and Section 7 concludes.

2 The Model

2.1 People

We begin by describing an environment in which fertility is chosen in a utility maximizing framework, in the tradition of Becker (1960), Razin and Ben-Zion (1975), and Becker and Barro (1988). The economy consists of \( N_t \) identical individuals, where \( t = 0, 1, 2, \ldots \) indexes time. Each individual obtains utility from consumption \( c_t \) and from the number of children \( b_t \) produced by the individual in period \( t \), according to

\[
u(c_t, b_t) = (1 - \mu) \frac{c_t^{1-\gamma}}{1-\gamma} + \mu \frac{b_t^{1-\eta}}{1-\eta},\]

(1)
where \( \bar{c}_t \equiv c_t - \bar{c} \) and \( \bar{b}_t \equiv b_t - \bar{b} \). The parameter \( \bar{c} > 0 \) denotes the subsistence level of consumption in this economy, and the parameter \( \bar{b} \geq 0 \) is related to the long-run rate of fertility, as we will see shortly.

We assume \( 0 < \mu < 1, \ 0 < \gamma < 1, \) and \( 0 < \eta < 1 \). These parameter restrictions ensure that the elasticity of substitution between consumption \( \bar{c} \) and children \( \bar{b} \) is always greater than one. This simple assumption will play an important role in generating the demographic transition.\(^2\)

Individuals are each endowed with one unit of labor per period, which they can use to obtain consumption or to produce children. Let \( \ell_t \) denote the amount of time the individual spends working, and let \( w_t \) denote the “wage” earned per unit of time worked. The technology for producing children is straightforward: each unit of time spent producing children leads to \( \alpha > \bar{b} \) births.

The individual’s optimization problem at each time \( t \) is given by

\[
\max_{c_t, b_t, \ell_t} u(c_t - \bar{c}, b_t - \bar{b})
\]

subject to

\[
c_t = w_t \ell_t
\]

and

\[
b_t = \alpha(1 - \ell_t),
\]

taking \( w_t \) as given. The fact that this optimization problem is static simplifies the analysis. This fact can be derived from a more general dynamic optimization problem under two assumptions. First, we assume that utility depends on the flow of births rather than on the stock of children. Second, we assume that the probability of death faced by an individual depends on aggregate per capita consumption, which individuals take as given. With

\(^2\)We do not necessarily require \( \eta < 1 \). Let \( z \equiv \frac{1-\mu}{\mu} \frac{1-\gamma}{\gamma-\eta} \). Then the elasticity of substitution between \( \bar{c} \) and \( \bar{b} \) is given by \( \frac{1+z}{\gamma z} \). It is constant when \( \gamma = \eta \) and takes the usual value \( 1/\gamma \).
these assumptions, the more standard dynamic optimization problem reduces to the sequence of static problems given above.

2.2 Production of the Consumption Good

The consumption good in this economy is produced using labor $L$, land $T$, and a stock of ideas $A$. Total output of this consumption good, denoted $Y$, is given by

$$Y_t = A_t^\sigma L_t^\beta T_t^{-1-\beta} \epsilon_t,$$

where $\sigma > 0$ and $0 < \beta < 1$, and $\epsilon_t$ is a productivity shock. This production function is assumed to exhibit constant returns to scale to the rivalrous inputs labor and land, and therefore increasing returns to labor, land, and knowledge taken together. As in Romer (1990), this assumption reflects a key property possessed by knowledge. Knowledge is nonrivalrous and can therefore be used at any scale of production without having to be reinvented.

The amount of land in this economy is fixed and normalized so that $T = 1$.

How is output divided to compensate the factor inputs? In this primitive economy, we do not assume the existence of a formal market structure. Instead, we assume that land is not owned by anyone and that total output is divided evenly among its labor, as might be the case in a society of hunters and gatherers:

$$w_t = Y_t / L_t = A_t^\sigma L_t^{\beta-1} \epsilon_t.$$  

2.3 Dynamics: Production of Ideas and People

The dynamics of this economy arise from two sources. First, people today produce knowledge that makes it easier to produce consumption goods in the future. Second, fertility and death lead to changes in the size of the population.

As above, $A_t$ denotes the stock of ideas at the start of period $t$. Therefore, $\Delta A_{t+1} \equiv A_{t+1} - A_t$ is the number of new ideas discovered during period $t$. 
In this economy, people produce new ideas according to
\[ \Delta A_{t+1} = \delta_t N_t^\lambda A_t^\phi, \] (7)
where \( \lambda > 0 \) and \( \phi < 1 \) are assumed. \( \delta_t \geq 0 \) is a stochastic shock to the production function for new ideas. The production of ideas is modeled very much like the production of any other good. Just as a larger labor force produces more widgets, a larger number of researchers produce more ideas. In this simplified formulation, neither human capital nor effort is required for ideas to be discovered, although both are obviously important in reality.

As in Jones (1995), the parameter \( \lambda \) allows for diminishing returns to changing the size of the population at a point in time. The parameter \( \phi \) allows the productivity of research to be either an increasing \( (\phi > 0) \) or a decreasing \( (\phi < 0) \) function of the stock of ideas that have been previously discovered.

The change in population is equal to the number of births minus the number of deaths:
\[ \Delta N_{t+1} = b_t N_t - d_t N_t \equiv n_t N_t, \quad N_0 > 0. \] (8)
The number of births per capita \( b_t \) is determined by the fertility behavior of individuals, discussed above. The mortality rate \( d_t \) is assumed to be a function of the average level of per capita consumption relative to subsistence, a useful summary measure of the technological capability of the economy as well as a measure that likely reflects the sensitivity of the population to disease and natural disasters. The mortality rate is given by
\[ d_t(c_t/\bar{c}) = f(c_t/\bar{c} - 1) + \bar{d}, \] (9)
where \( f(\cdot) \) is some decreasing function such that \( f(0) > 1 + \alpha \) and \( f(\infty) = 0 \). As consumption rises, the mortality rate falls. As per capita consumption falls to the subsistence level, everyone in the population dies. This characteristic implicitly defines what we mean by subsistence. Notice also that \( \bar{d} \geq 0 \) denotes the mortality rate in an economy with infinitely large consumption.
2.4 Equilibrium

The setup of the economy is now complete and we can define the equilibrium.

**Definition:** A *static equilibrium* in this economy in period $t$ is a collection of allocations and prices $(c_t, \ell_t, Y_t, L_t, b_t, w_t)$ such that, given values of the state variables $A_t$ and $N_t$ and a productivity shock $\epsilon_t$, the choice variables $c_t$, $b_t$, and $\ell_t$ solve the representative individual's maximization problem and the wage $w_t$ is given by equation (6), where $L_t \equiv \ell_t N_t$.

**Definition:** A *dynamic equilibrium* in this economy is a sequence of static equilibrium allocations for $t = 0, 1, 2, \ldots$, together with sequences for $\{A_t, N_t, d_t, n_t\}_{t=0}^{\infty}$, such that, given an exogenous sequence of shocks $\{\delta_t, \epsilon_t\}_{t=0}^{\infty}$ and given the initial conditions $A_0$ and $N_0$, the evolution of the economy satisfies the laws of motion in equations (7), (8), and (9) and the constraints $A_t \geq 0$ and $N_t \geq 0$.

Solving for the equilibrium is straightforward. The individual's maximization problem yields the first order condition

$$\frac{u_b}{u_c} = \frac{w_t}{\alpha}. \quad (10)$$

This says that an individual must be indifferent at the optimum between spending a little more time working and spending a little more time producing children.

With the preferences given by equation (1), this first order condition implies

$$\tilde{b}_t = \left(\frac{\alpha \mu \tilde{c}_t^\gamma}{1 - \mu w_t}\right)^{1/\eta}. \quad (11)$$

Along a balanced growth path in this model, $\tilde{c}_t$ and $w_t$ will grow at the same rate. The assumption that $0 < \gamma < 1$, then, is what allows this model to exhibit a fertility transition, i.e. a situation in which fertility eventually declines as the wage rate rises.

This can be seen formally by noting that equations (3) and (4) imply a second relationship $c_t = w_t(1 - b_t/\alpha)$. Substituting this expression for $c_t$
into (11) one gets an implicit expression for $\tilde{b}_t$ as a function of the wage $w_t$. Differentiating this expression, the sign of $\frac{db}{dw}$ is the same as the sign of

$$\frac{\bar{c}}{\gamma}w_t - \left(\frac{1}{\gamma} - 1\right)(1 - \frac{\bar{b}_t}{\alpha}).$$

The traditional income and substitution effects are reflected in the second term. As the wage goes up, the income effect leads individuals to increase both consumption and fertility. The substitution effect, on the other hand, leads people to substitute toward consumption and away from fertility: the discovery of new ideas raises the productivity of labor at producing consumption, but the technology for producing children is unchanged. If $\gamma < 1$, then the substitution effect dominates, while if $\gamma > 1$, the income effect dominates. As usual, if $\gamma = 1$, i.e. with log utility, these two effects offset.

A third effect not traditionally present is reflected in the first term: as the wage rises, the subsistence consumption level which the consumer is required to purchase gets cheaper, leading consumers to have more after-subsistence income to spend on both more children and more consumption. This effect disappears as the wage gets large. The assumption that $0 < \gamma < 1$, then, leads the subsistence effect to dominate for small values of the wage and the substitution effect to dominate for large values of the wage, producing one component of the demographic transition: fertility rises and then falls as the wage rate rises.

Based on this first order condition, the following proposition (proved in the appendix) establishes a simple condition under which an interior static equilibrium exists and is unique.

**Proposition:** Let $a_t \equiv A_t^\alpha \epsilon_t / N_t^{1-\beta}$ be a measure of productivity in this economy. Assume this measure of productivity is sufficiently large that $(\bar{c}/a_t)^{1/\beta} < 1 - \bar{b}/\alpha$. Then, there exists a unique interior static equilibrium $\ell^*(a_t), w^*(a_t), c^*(a_t),$ and $b^*(a_t)$.

The technical condition in the proposition is needed for an interior solution. In the case in which this condition is just violated, $(\bar{c}/a_t)^{1/\beta} =$
Growth Over the Very Long Run

\[ 1 - \frac{b}{\alpha} = \ell^* \]

The population is so large relative to the technology level that diminishing returns to land reduces the wage leading to \( c^* = \bar{c} \) and \( b^* = \bar{b} \). Given the mortality function in equation (9), everyone in the economy would die in that period, and the population would be zero from then on.

3 Dynamics and Stability

To see how the static equilibrium evolves over time, we proceed as follows. First, we shut down the shocks in the model: we assume for the moment that \( \delta_t = \delta > 0 \) and \( \epsilon_t = 1 \) for all \( t \). Then, we characterize the equilibrium along a steady-state balanced growth path. Finally, we explore the dynamics and the stability properties of this path.

3.1 Balanced Growth

A balanced growth path is a situation in which all variables grow at constant geometric rates (possibly zero). We will look for a balanced growth path in which \( \ell, b, \) and \( d \) are constant. To characterize the balanced growth path of this economy, begin with the production function for new ideas, equation (7). In the absence of technology shocks, this equation implies

\[ A_{t+1} = N t A_1 - t : (12) \]

Along a balanced growth path, the left-hand-side of this equation is constant by definition, so the right-hand-side must also be constant. This is true when

\[ G_A = G_{N^{\lambda - \sigma}} \]

where \( G_z \) is defined as the gross growth rate of any variable \( z \) along a balanced growth path, i.e. \( G_z \equiv \frac{z_{t+1}}{z_t} \).

Along a balanced growth path, \( \ell \) is constant. Since \( c_t = w_t \ell_t \) and \( w_t = a_t / \ell_t^{1-\beta} \), consumption is given by \( c_t = a_t \ell_t^\beta \). Finally, note that with \( \epsilon_t = 1 \),
where \( a_t = A_t^\sigma / N_t^{1-\beta} \). Therefore, along a balanced growth path,

\[
G_c = G_w = G_a = \frac{G^\sigma_A}{G^\theta_N} = G^\theta_N,
\]

(14)

where \( \theta = \frac{\lambda \sigma}{1-\phi} - (1 - \beta) \). Clearly, sustained per capita growth requires \( \theta > 0 \), which we will now assume. As in several recent papers, the growth rate of per capita income and consumption along the balanced growth path is proportional to the rate of population growth.

The assumption of \( \theta > 0 \) implies that the model is characterized by increasing returns to accumulable factors. For example, suppose that the production of ideas is homogeneous of degree one, so that \( \lambda + \phi = 1 \). It is easy to show that \( \theta > 0 \) then requires \( \sigma + \beta > 1 \). Recall that the nonrivalry of ideas motivated the assumption of constant returns to land and labor and increasing returns to land, labor, and ideas together — i.e. \( \sigma > 0 \). We require the stronger assumption that there are increasing returns to ideas and labor, holding land constant. That is, the increasing returns implied by nonrivalry must be sufficiently strong.

Turning to population growth, notice that

\[
\frac{\Delta N_{t+1}}{N_t} = n(a_t) = b(a_t) - d(a_t),
\]

(15)

where we abuse notation somewhat in writing \( d(a_t) \) for \( d(c(a_t)/\bar{c}) \). In addition, \( b(a_t) = \alpha (1 - \ell(a_t)) \), so that a constant \( \ell \) requires constant birth and mortality rates along the balanced growth path. Under what condition will these rates be constant? Recall that the first order condition for the individual’s optimization problem in equation (11) leads her to set the excess birth rate \( \tilde{b}_t \) proportional to \( (\tilde{c}_t^\gamma / w_t)^{1/\eta} \). Therefore, we need \( \tilde{c}_t^\gamma / w_t \) to be constant. Along a balanced growth path, however, \( c_t \) and \( w_t \) grow at the same rate. This implies that a balanced growth path occurs only asymptotically as \( c_t \) and \( w_t \) go to infinity, and the demographic transition effects associated with \( \gamma < 1 \) apply. As this happens, \( \tilde{b}_t \) approaches zero so that \( b_t \) approaches \( \bar{b} \).
In addition, the mortality rate approaches $\bar{d}$. We assume that $\bar{b} \geq \bar{d}$ so that

$$G_N = 1 + \bar{b} - \bar{d} \geq 1.$$  \hspace{1cm} (16)

Applying this result to equation (14), we see that along the balanced growth path,

$$G_c = G_w = G_a = G_{Y/N} = (1 + \bar{b} - \bar{d})^\beta.$$ \hspace{1cm} (17)

Notice that increasing returns is not sufficient for positive per capita growth along the balanced growth path. If $\bar{b} = \bar{d}$, then there is no population growth in the long run and the balanced growth path has zero per capita growth.

One must also be careful with the asymptotic nature of this result. The balanced growth path in this model is an asymptotic result that applies only as $a_t$ goes to infinity. For example, even if the balanced growth path has zero (geometric) per capita growth, the growth rate of per capita income will be positive in every period and the level of per capita income will go to infinity.\(^3\)

### 3.2 The Demographic Transition

A necessary prelude to characterizing the stability of the balanced growth path and the nature of the transition dynamics of the model is an analysis of population growth and the demographic transition. Recall that $n(a) = b(a) - d(a)$. We examine $b(a)$ and $d(a)$ in turn.

First, $b(a) = a(1 - \ell(a))$. And $\ell(a)$ is the solution of the following nonlinear equation, obtained by combining (25) and $w = a/\ell^{1-\beta}$:

$$F(\ell) \equiv a(\alpha(1 - \ell) - \bar{b})^n - \frac{\alpha \mu}{1 - \mu} (a\ell^\beta - \bar{c})^\gamma \ell^{1-\beta} = 0.$$ \hspace{1cm} (18)

\(^3\)It is interesting to compare the result in equation (17) to that in Jones (1998). Unlike the results in that paper, this model, even with endogenous population growth, has the character of a semi-endogenous growth model: the long-run growth rate depends only on exogenous parameters that are unlikely to be changed by policy. The source of this difference is that the elasticity of substitution in the utility function between consumption and offspring is not forced to equal one. The model here therefore highlights the assumption of a unitary elasticity of substitution as crucial to the policy results in that paper.
Totally differentiating both sides of this equation with respect to $a$ and $\ell$, one sees that

$$\text{sign} \left\{ \frac{d\ell}{da} \right\} = \text{sign} \left\{ 1 - \frac{\gamma}{1 - \frac{c}{\ell}} \right\},$$

where $c = a\ell(a)^\beta$. The conditions that $\tilde{c} > 0$ and $\tilde{b} > 0$ limit the range of values that $\ell(a)$ can take to $(\tilde{c}/a)^{1/\beta} < \ell < 1 - \tilde{b}/\alpha$. When $a = a^D \equiv \frac{\tilde{c}}{(1 - \tilde{b}/\alpha)^\beta}$, this range shrinks to the single point at which $\ell = (\tilde{c}/a^D)^{1/\beta} = 1 - \tilde{b}/\alpha$. On the other hand, we have already shown that $\lim_{a \to \infty} \ell(a) = 1 - \tilde{b}/\alpha$.

These endpoint conditions, together with the conditions on the slope given by equation (19) imply that the solution $\ell(a)$ has the shape given in Figure 1.

Given the solution for $\ell(a)$, $b(a)$ is simply $\alpha(1 - \ell(a))$, as shown in the right panel of Figure 1. Also shown in this figure is a $d(a)$ schedule; it is easy to show that $c(a)$ is a monotonic function, so that $d(a)$ has the same general shape as $d(c)$. In addition to the assumptions already made about $d(a)$, we assume that $f(\cdot)$ is such that the point $a^M$ where $b(a^M) = d(a^M)$ occurs at a value of $a$ less than that which maximizes the birth rate, and we assume that this is the only intersection of $b(a)$ and $d(a)$.

Finally, provided the function $f(\cdot)$ is restricted appropriately, we can
now characterize $n(a) \equiv b(a) - d(a)$ as shown in Figure 2. The population growth rate is zero when productivity is $a^M$. It increases as a function of $a$ to a level greater than $b - d$ (at least for a range of parameters values), and then declines to its balanced growth path level as $a$ goes to infinity. The demographic transition is apparent in both Figures 1 and 2.

This general picture describes the classic version of the demographic transition. As summarized by Cohen (1995) and Easterlin (1996), the demographic transition consists of two phases. In the first, called a mortality revolution, mortality rates fall sharply, driven by advances in health technology. Birth rates either remain relatively constant or perhaps even rise slightly. The result is an increase in population growth rates. The second phase is the fertility revolution, characterized by a birth rate that now falls more quickly than the relatively low but still declining mortality rate. The result is a decline in population growth rates.
3.3 Stability and Transition Dynamics

To analyze the stability properties of this economy, it is helpful to consider two state-like variables, the productivity variable $a_t$ and a second variable $x_t \equiv \delta N_t^\lambda / A_t^{1-\phi}$. That is, $x_t$ is the growth rate of $A_t$. The dynamics of $x_t$ are given by

$$\frac{x_{t+1}}{x_t} = (1 + n(a_t))^{\lambda} / (1 + x_t)^{1-\phi}. \quad (20)$$

Since $a_t = A_t^\sigma / N_t^{1-\beta}$ (again with $\epsilon_t = 1$), the dynamics of this state-like variable are given by

$$\frac{a_{t+1}}{a_t} = (1 + x_t)^{\sigma} / (1 + n(a_t))^{1-\beta}. \quad (21)$$

In addition, $x_0$ and $a_0$ are given by the initial conditions on $A_0$ and $N_0$.

These two equations, together with the static equilibrium conditions that determine $n(a_t)$, completely characterize the dynamics of the economy. These dynamics can be examined in the discrete time version of a phase diagram. Notice that the balanced growth path occurs when $\Delta x_t = 0$ and $a_t = \infty$, so that the analysis of this system is slightly different from the traditional phase diagram analysis. Under the increasing returns assumption that $\theta > 0$, the $\Delta x_t = 0$ schedule lies “above” the $\Delta a_t = 0$ schedule, and the dynamics are characterized as in Figure 3.

As drawn in the figure, the dynamics are quite rich. The balanced growth path is globally stable: if the economy begins at any point such that $x_t > 0$ and $a_t > a^D$, it converges to the balanced growth path. However, we see that the growth rate of $A_t$ will not generally be monotonically along the transition to the balanced growth path. It is natural to think of the economy starting from a point with a low $a$ — low consumption — and a low $x$ — slow technological progress. Then, the general pattern is for growth rates to rise and then fall as the economy approaches the balanced growth path.

It is also easy to see the importance of the increasing returns assumption that $\theta > 0$. If $\theta = 0$, so that the economy is characterized by constant

---

4For this purpose, it is helpful to think of a period as being a brief interval of time.
returns to accumulable factors, then the $\Delta x_t = 0$ and $\Delta a_t = 0$ curves lie on top of each other. In this case, the dynamics of the economy move it toward this curve, and the economy tends toward a situation in which $a_t$ and $x_t$ — and therefore per capita consumption — are constant.\footnote{One has to be careful here. Because time is discrete, the economy could cycle around such a point.} If $\theta < 0$, the $\Delta a_t = 0$ schedule lies “above” the $\Delta x_t = 0$ schedule. In this case, there exists a globally stable steady state at the point $a_t = a^M$. Population growth (eventually) falls toward zero and consumption falls to some Malthusian-style subsistence level. In both of these cases, technological progress occurs forever, and the population grows to infinity. However, the key point is that technological progress does not translate into growth in per capita consumption, and the economy never experiences an industrial revolution.

Returning again to the $\theta > 0$ case, we are ready to consider what happens if shocks are added back into the system. Both the level of $x$ and the level of $a$ can jump as a result of shocks. A positive idea shock causes the level of $x$ to jump upward. A productivity shock that reduces $\epsilon_t$ causes $a_t$ to decline. However, apart from these jumps, the dynamics of the economy are still determined as in Figure 3.

Finally, consider the effect of an exogenous increase in mortality, such as
the Black Death in 14th century Europe. Such a shock reduces $N$, causing $a$ to jump to the right and $x$ to jump down. The result is a rise in the level of the wage and the level of consumption in the short run, and a reduction in the rate of technological progress. As discussed by Lee (1988), population and per capita consumption can be negatively related in the short run even though they are positively related in the long run.

4 Quantitative Analysis

4.1 The “Facts”

The model developed in the previous section will be analyzed quantitatively to help us understand growth over the very long run in both population and per capita income. First, however, we pause to present the “facts” about these two variables.

Cohen (1995) assembles data on world population from a number of studies conducted during the last forty years and provides a brief overview of the data. McEvedy and Jones (1978), the main source used by Kremer (1993), appears to be the most thorough study, and I will rely on Kremer’s data, as reported in Table 1.

It is useful to appreciate both the extremely low rate of population growth over most of history as well as the time scale over which this rate operates. For example, using Kremer’s collection of world population data, the rate of population growth, measured as the average annual change in log population, was only 0.0000072 between 1 million B.C. and 1 A.D. Nevertheless, over this period, the level of population increased by a factor of 1360: from 0.125 million people in 1 million B.C. to 170 million people in 1 A.D. A second key fact about population growth apparent in the table, emphasized by Kremer (1993), is that the rate of population growth is itself generally increasing over time. This is true not only in recent centuries but also dating back to our earliest data.
Table 1: Population Data

<table>
<thead>
<tr>
<th>Year (Millions)</th>
<th>Population Level (Millions)</th>
<th>Average Annual Growth Rate over Preceding Period</th>
</tr>
</thead>
<tbody>
<tr>
<td>-25000</td>
<td>3.34</td>
<td>...</td>
</tr>
<tr>
<td>-10000</td>
<td>4</td>
<td>0.000012</td>
</tr>
<tr>
<td>-5000</td>
<td>5</td>
<td>0.000045</td>
</tr>
<tr>
<td>-4000</td>
<td>7</td>
<td>0.000336</td>
</tr>
<tr>
<td>-3000</td>
<td>14</td>
<td>0.000693</td>
</tr>
<tr>
<td>-2000</td>
<td>27</td>
<td>0.000657</td>
</tr>
<tr>
<td>-1000</td>
<td>50</td>
<td>0.000616</td>
</tr>
<tr>
<td>-500</td>
<td>100</td>
<td>0.001386</td>
</tr>
<tr>
<td>-200</td>
<td>150</td>
<td>0.001352</td>
</tr>
<tr>
<td>0</td>
<td>170</td>
<td>0.000626</td>
</tr>
<tr>
<td>200</td>
<td>190</td>
<td>0.000556</td>
</tr>
<tr>
<td>400</td>
<td>190</td>
<td>0.000000</td>
</tr>
<tr>
<td>600</td>
<td>200</td>
<td>0.000256</td>
</tr>
<tr>
<td>800</td>
<td>220</td>
<td>0.000477</td>
</tr>
<tr>
<td>1000</td>
<td>265</td>
<td>0.000931</td>
</tr>
<tr>
<td>1100</td>
<td>320</td>
<td>0.001886</td>
</tr>
<tr>
<td>1200</td>
<td>360</td>
<td>0.001178</td>
</tr>
<tr>
<td>1300</td>
<td>360</td>
<td>0.000000</td>
</tr>
<tr>
<td>1400</td>
<td>350</td>
<td>-0.000282</td>
</tr>
<tr>
<td>1500</td>
<td>425</td>
<td>0.001942</td>
</tr>
<tr>
<td>1600</td>
<td>545</td>
<td>0.002487</td>
</tr>
<tr>
<td>1700</td>
<td>610</td>
<td>0.001127</td>
</tr>
<tr>
<td>1800</td>
<td>900</td>
<td>0.003889</td>
</tr>
<tr>
<td>1900</td>
<td>1625</td>
<td>0.005909</td>
</tr>
<tr>
<td>2000</td>
<td>5333</td>
<td>0.011884</td>
</tr>
</tbody>
</table>

Note: The levels of population are taken from Kremer (1993), who in turn takes his data from various sources. The population growth rate is computed as the average annual change in the natural log of population over the preceding interval. Two changes relative to Kremer are made. First, the year 1 A.D. is set equal to the year 0. Second, the population in 1990 is used for the population in the year 2000. These changes are made so that the period length in the model can be set equal to 25 years. The growth rates for a few periods are slightly different from those in Kremer because he reports growth rates from his underlying sources rather than based on the levels themselves.
Data on per capita GDP or per capita consumption are much harder to come by. Nevertheless, the collection of evidence seems to support the following stylized picture: there was relatively little net increase in standards of living over most of history, say prior to the year 1500. Since then, per capita growth has risen, and levels of per capita income are now substantially higher than they were prior to 1500.

For example, Maddison (1982) estimates zero per capita income growth in Europe between 500 and 1500. Lee (1980) finds that the real wage in England in 1800 was nearly unchanged from its level in 1300; Hansen and Prescott (1998) make use of some new data assembled by Gregory Clark to draw a similar conclusion. Jevons (1896) uses detailed wage records from Athens in 328 B.C. to argue that wages in ancient Greece were roughly the same as those in Britain in the 15th century.\footnote{Curiously, Clark (1940, p. 164ff) takes this calculation further to argue that the same statement is true of “modern” Britain, i.e. apparently in the 1920s or 1930s.}

Regarding the level of world GDP per capita, Maddison (1995) reports an estimate of $565 (in 1990 dollars) for the year 1500 and $5145 in 1992. De Long (1998) reports values ranging from $115 to $512 in 1500 and $5204 in 1990, depending on whether or not an admittedly-course correction is made for quality change.

Growth rates of world per capita GDP can be computed from Maddison (1995, Table G-3). The average growth rate from 1820 to 1900 was 0.83 percent per year, and that from 1900 to 1990 was 1.57 percent per year. Using decadal averages, world per capita GDP growth peaked in the 1960s at 3.00 percent per year before falling to 1.85 percent in the 1970s and 1.38 percent in the 1980s.

### 4.2 Parameter Choices

To simulate the model, values for 16 parameters are required. We will fix some of the parameter values ahead of time and then estimate some others to
Table 2: Baseline Parameter Values

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
<th>Comment</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\bar{c}$</td>
<td>50</td>
<td>Death with probability 1 at $c = \bar{c}$</td>
</tr>
<tr>
<td>$N_0$</td>
<td>3.34</td>
<td>Population (in millions) in year 25000 B.C.</td>
</tr>
<tr>
<td>$\beta$</td>
<td>2/3</td>
<td>Land share = 1/3 (Kremer)</td>
</tr>
<tr>
<td>$\alpha$</td>
<td>1/10</td>
<td>Maximum fertility rate per period</td>
</tr>
<tr>
<td>$\bar{b}$</td>
<td>0</td>
<td>Long-run fertility rate</td>
</tr>
<tr>
<td>$\bar{d}$</td>
<td>0</td>
<td>Long-run mortality rate</td>
</tr>
<tr>
<td>$\sigma$</td>
<td>1</td>
<td>Unidentified</td>
</tr>
<tr>
<td>$A_0$</td>
<td>648.45</td>
<td>To produce $n_0 = 0$ in 25000 B.C.</td>
</tr>
<tr>
<td>$\lambda$</td>
<td>3/4</td>
<td>Duplication of research</td>
</tr>
<tr>
<td>Period Length</td>
<td>25</td>
<td>Period length of 25 years</td>
</tr>
</tbody>
</table>

Note: Parameters of preferences and the birth and death technologies are estimated to fit birth and mortality rates, as described in the text.

fit the population data as well as possible. One issue that arises immediately is the distinction between consumption and GDP. This distinction is not present in the model, and we will choose for convenience to match up $c$ in the model with data on per capita GDP.

The parameter values that are fixed in advance are summarized in Table 2. The parameter $\bar{c}$ is set equal to 50, measured in 1990 dollars. If per capita consumption were to fall to $\bar{c} = 50$, everyone in the economy would die immediately. The parameter $N_0$ is set equal to 3.34, corresponding to the world population (in millions) in the year 25000 B.C., the first year of our simulation.

The parameter $\beta$ is set equal to 2/3, so that the land share in an economy with perfect competition and property rights would be 1/3. This is the value chosen by Kremer (1993). The parameter $\alpha$ corresponds to the maximum birth rate at any instant in time, and we set the value of this parameter to 1/10. Easterlin (1996) and Livi-Bacci (1997, p. 7) report that maximum birth rates over history are about 0.05. With $\alpha = 1/10$, this birth rate
occurs when one half of the individual’s labor endowment is devoted entirely to raising children.

The parameters $\bar{b}$ and $\bar{d}$ correspond to the asymptotic birth rate and mortality rate in the model (that is, as consumption goes to infinity). How many kids would people like to have when consumption is infinite? We assume $\bar{b} = 0$, although one could make a case that this number should be positive. We also assume that the mortality rate goes to zero, so that people eventually can live forever. These assumptions imply that population growth goes to zero as consumption gets large.

For the elasticity of output with respect to new ideas, we set $\sigma = 1$. There always exists a value of $\phi$ consistent with any value of $\sigma > 0$ such that the model produces observationally equivalent results for population, consumption, and total factor productivity $A^\sigma$. These two parameters could be distinguished with data on the stock of ideas, but absent this data, they cannot be distinguished. We set $\sigma = 1$ so that we measure ideas in units of total factor productivity. The parameter $\phi$, then, is conditional on this value, and would change (in a predictable fashion) for other values of $\sigma$.

We assume a period length of 25 years, so that data on population are only observed infrequently. The initial condition $A_0$ is chosen so that the population growth rate in the first 25-year period is equal to 0, given all of the values for the other parameters. This leads to a value of $A_0 = 648.45$.

The final assignment in Table 2 sets $\lambda$ equal to $3/4$. If the population were instantaneously doubled, one suspects that the number of new ideas discovered would increase by less than a factor of two because the same idea would likely be discovered multiple times. This suggests an elasticity less than one. Choosing a specific value for $\lambda$ is more difficult. We discuss below an unsuccessful attempt to estimate $\lambda$ using the population data. Jones and Williams (1996) suggest that a value of $3/4$ seems reasonable based on estimates of social rates of return. In the simulations below, this value produces plausible results.
Table 3: Observations on Mortality Rates

<table>
<thead>
<tr>
<th>Per capita Consumption</th>
<th>“Data” Mortality Rate</th>
<th>Fitted Mortality Rate</th>
<th>Comments</th>
</tr>
</thead>
<tbody>
<tr>
<td>100</td>
<td>.053</td>
<td>.053</td>
<td>Livi-Bacci (1997)</td>
</tr>
<tr>
<td>250</td>
<td>.05</td>
<td>.051</td>
<td></td>
</tr>
<tr>
<td>800</td>
<td>.04</td>
<td>.038</td>
<td></td>
</tr>
<tr>
<td>5000</td>
<td>.01</td>
<td>.011</td>
<td>Cohen (1995) for 1985-90</td>
</tr>
<tr>
<td>20000</td>
<td>.007</td>
<td>.003</td>
<td>Canada, 1989</td>
</tr>
<tr>
<td>100000</td>
<td>.001</td>
<td>.001</td>
<td></td>
</tr>
</tbody>
</table>

Notes: The second and third data points are taken from rough guesses by Livi-Bacci (1997) (p. 7) that average mortality rates range as high as four to five percent and makes an educated guess that the mortality rate averaged something like four percent between 1 A.D. and 1750. The consumption numbers corresponding to these observations are simple guesses that seem plausible given the analysis of Pritchett (1997) on minimum income levels. The next two data points are taken from Cohen (1995, p. 68). Based on Maddison (1995), I assume that these years correspond to per capita GDPs of 2000 and 5000 dollars. Finally, the last number corresponds to the mortality rate in Canada in 1989 according to the World Bank (1991), Table 27. The mortality rate for the United States in that year was slightly higher, at 0.9 percent, while in Japan, Hong Kong, and Australia it was lower.

The remaining parameter values are estimated in two stages. First, we estimate the parameters of the mortality function to fit some very rough statistics. Recall that the mortality function is given by

\[
d_t(z_t) = f(z_t) + \bar{d}, \quad z_t \equiv c_t/\bar{c} - 1.
\]  

We assume that \( f(z) \) is the reciprocal of a polynomial: \( f(z) = 1/(\omega_1 z^{\omega_2} + \omega_3 z) \). We then estimate the \( \omega_i \) parameters using nonlinear least squares to fit the observations given in Table 3.

The last column of numbers in Table 3 reports the fitted mortality rates;
the coefficients themselves are reported in Table 4. The equation fits quite well, with an $R^2$ of 0.989.

Given the mortality function $d(c)$, we turn to estimating the parameters related to fertility. As with death rates, we have very little information upon which to base our estimates of these parameters. The observations we draw on are given in Table 5 and describe population growth and consumption. The first and the last two observations in this table are rough guesses. Motivated in part by Pritchett (1997), we assume that the Malthu-
The consumption level at which births and deaths are equalized is something like two hundred fifty dollars (in 1990 dollars). The next to last observation corresponds roughly to per capita income and population growth in the richest countries today. The intermediate observations are taken from two sources. The consumption levels correspond to the world per capita GDP levels reported by Maddison (1995) in Table G-3.\footnote{Maddison does not report a value for 1875. I use the interpolated value from De Long (1998).} The population growth rates, corresponding to the years reported in the comment, are taken from Kremer (1993).

Given the mortality function \(d(c)\) that we have already estimated, and given \(\alpha = .1\), we estimate \(\mu, \gamma,\) and \(\eta\) to fit the population growth rate data in Table 5 as well as possible. Specifically, we estimate these parameters using nonlinear least squares to minimize the sum of squared deviations between the observed population growth rate and the model’s predicted population growth rate at the given levels of consumption. The results of this estimation are reported in Table 6. Figure 4 plots the birth and mortality functions, together with the population growth rates that these functions imply.

From the fitted values in Table 5 and from the figures, one sees that this

### Table 6: Parameter Estimates of the Utility Function

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Coefficient</th>
<th>Standard Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\mu)</td>
<td>.9844</td>
<td>.0012</td>
</tr>
<tr>
<td>(\gamma)</td>
<td>.8141</td>
<td>.0164</td>
</tr>
<tr>
<td>(\eta)</td>
<td>.0250</td>
<td>.0299</td>
</tr>
<tr>
<td>(R^2)</td>
<td>.969</td>
<td></td>
</tr>
</tbody>
</table>
simple fertility model performs well. The model generates a demographic transition and broadly matches the features of the data.

4.3 Simulating the Model without Shocks

Figures 5 and 6 report the results from simulating the model with the parameter values obtained in the previous section in the absence of shocks. For this simulation, we set $\delta = .00104$ and $\phi = 3/4$. These parameter values will be discussed further in the next section; for the moment, just take them as an example.

Figure 5 reports average annual growth rates for the data on population (the circles) as well as the model’s simulated growth rates for population and per capita consumption. Figure 6 displays the level of population and the level of per capita consumption, both in the simulation and for the data.

These figures illustrate that the internal dynamics of the model are able to replicate broadly the patterns observed in the data. The simulation exhibits thousands of years of very slow growth, followed by a sharp rise around

---

8In this and subsequent figures, the circles will be used to denote the data discussed in the previous section.
the time of the Industrial Revolution. In levels, the model systematically
overpredicts the level of population but does an excellent job of matching
the data on per capita consumption. In particular, the level of consump-
tion is stable for thousands of years before rising sharply with the Industrial Revolution.

The future predictions of the model are interesting even if they should not be taken seriously. With $\bar{b} = \bar{d} = 0$, the long run growth rate of the model is zero. Population growth falls to zero after the Industrial Revolution, as the rise in consumption generates a demographic transition. Consumption growth falls to zero gradually, but only after the level of per capita consumption is well on its way toward infinity.

5 Adding Shocks to the Model

Recall the production function for new ideas in equation (7), written in a slightly different form here:

$$\Delta A_{t+1} = \tilde{\delta}_t N_t^\lambda,$$

(23)

where $\tilde{\delta}_t = \delta_t A_t^\phi$ is a stochastic process. Obviously, any stochastic equilibrium based on the idea production function in equation (7) with $\phi$ general can be represented as an equilibrium in the model with the idea production function in equation (23) given the definition of $\tilde{\delta}$. Therefore, as a first pass, we can focus on this second version of the model and then later decide how to decompose $\tilde{\delta}_t$ into the true shocks $\delta_t$ and the “spillover” portion of the production function for ideas.\(^9\)

We are now ready to outline the solution for the model. Basically, we solve for a sequence of idea shocks $\tilde{\delta}_t$ and TFP shocks $\epsilon_t$, so that the model’s simulated levels of population exactly match the actual levels. Both shocks are assumed to be constant during the entire interval between successive observations on the level of population (recall that a period is only 25 years

\(^9\)There is one minor problem with this reasoning in practice, given the solution technique outlined in the next paragraph. We solve for the idea shocks assuming that the shock is constant between observations of population data. Clearly, however, $\delta_t$ and $\delta_t$ cannot both be constant. However, the approximation is close enough to be useful as a starting point.
and that we therefore observe the level of population infrequently). If a positive idea shock works, we shut o the TFP shocks for that interval \((\epsilon = 1)\). On the other hand, if the level of population declines or grows very slowly, it is possible that even a constant stock of ideas will overpredict the subsequent level of population. In this case, we set the idea shock equal to zero to produce the constant stock of ideas over the interval. We then find the value of \(\epsilon < 1\) such that the subsequent level of population is matched exactly. In this sense, we use “positive” idea shocks and “negative” TFP shocks to match the population data exactly. The solution method is provided in more detail in the appendix.

Ideally, one would of course like to allow for a richer specification for the shocks | for example, by allowing for TFP shocks even when the idea shock is positive. Given the limited nature of the data, however, it does not seem possible to identify the shocks in such a specification.

Solving the model in this fashion, given the population data in Table 1, leads to the sequence of idea shocks and TFP shocks reported in Table 7. Each type of shock will be discussed in turn.

With the normalization \(\sigma = 1\), the stock of ideas is measured in standard total factor productivity units. Holding constant the level of the labor force, a one percentage point change in the stock of ideas raises output and per capita consumption by one percentage point. Over long periods of time however, the labor force is not constant, and because land is present in fixed supply, the expansion of the labor force tends to reduce output, other things equal. In other words, it requires a larger increase in the stock of ideas to raise per capita consumption by one percentage point today than it did five thousand years ago.

With this in mind, we also report the idea shocks in output units. That is, consider the following partial difference:

\[
\Delta y_{t+1} = \frac{\Delta A_{t+1}}{L_{t}^{1-\beta}} = \frac{\tilde{\delta}_{t} N_{t}^{\lambda}}{L_{t}^{1-\beta}} = \tilde{\rho}_{t} N_{t}^{\lambda},
\]  

(24)
Table 7: Productivity Shocks: $\tilde{\epsilon}_t$ and $\epsilon_t$

<table>
<thead>
<tr>
<th>Year</th>
<th>$N_t$</th>
<th>$\delta_t$</th>
<th>$\epsilon_t$</th>
<th>$\hat{\delta}_t$</th>
<th>$\hat{\epsilon}_t$</th>
</tr>
</thead>
<tbody>
<tr>
<td>-2500</td>
<td>3.34</td>
<td>0.026</td>
<td>0.012</td>
<td>1</td>
<td>0.064</td>
</tr>
<tr>
<td>-1000</td>
<td>4</td>
<td>0.093</td>
<td>0.042</td>
<td>1</td>
<td>0.217</td>
</tr>
<tr>
<td>-5000</td>
<td>5</td>
<td>0.966</td>
<td>0.436</td>
<td>1</td>
<td>2.065</td>
</tr>
<tr>
<td>-4000</td>
<td>7</td>
<td>1.317</td>
<td>0.595</td>
<td>1</td>
<td>2.368</td>
</tr>
<tr>
<td>-3000</td>
<td>14</td>
<td>0.636</td>
<td>0.287</td>
<td>1</td>
<td>0.915</td>
</tr>
<tr>
<td>-2000</td>
<td>27</td>
<td>0.565</td>
<td>0.255</td>
<td>1</td>
<td>0.742</td>
</tr>
<tr>
<td>-1000</td>
<td>50</td>
<td>1.903</td>
<td>0.860</td>
<td>1</td>
<td>1.996</td>
</tr>
<tr>
<td>-500</td>
<td>100</td>
<td>0.000</td>
<td>0.000</td>
<td>0.986</td>
<td>0.000</td>
</tr>
<tr>
<td>-200</td>
<td>150</td>
<td>0.000</td>
<td>0.000</td>
<td>0.971</td>
<td>0.000</td>
</tr>
<tr>
<td>0</td>
<td>170</td>
<td>0.000</td>
<td>0.000</td>
<td>0.942</td>
<td>0.000</td>
</tr>
<tr>
<td>200</td>
<td>190</td>
<td>0.000</td>
<td>0.000</td>
<td>0.985</td>
<td>0.000</td>
</tr>
<tr>
<td>400</td>
<td>190</td>
<td>0.000</td>
<td>0.000</td>
<td>0.971</td>
<td>0.000</td>
</tr>
<tr>
<td>600</td>
<td>200</td>
<td>0.527</td>
<td>0.238</td>
<td>1</td>
<td>0.289</td>
</tr>
<tr>
<td>800</td>
<td>220</td>
<td>0.888</td>
<td>0.401</td>
<td>1</td>
<td>0.864</td>
</tr>
<tr>
<td>1000</td>
<td>265</td>
<td>3.863</td>
<td>1.745</td>
<td>1</td>
<td>2.823</td>
</tr>
<tr>
<td>1100</td>
<td>320</td>
<td>0.000</td>
<td>0.000</td>
<td>0.814</td>
<td>0.000</td>
</tr>
<tr>
<td>1200</td>
<td>360</td>
<td>0.000</td>
<td>0.000</td>
<td>0.707</td>
<td>0.000</td>
</tr>
<tr>
<td>1300</td>
<td>360</td>
<td>0.000</td>
<td>0.000</td>
<td>0.676</td>
<td>0.000</td>
</tr>
<tr>
<td>1400</td>
<td>350</td>
<td>0.000</td>
<td>0.000</td>
<td>0.939</td>
<td>0.000</td>
</tr>
<tr>
<td>1500</td>
<td>425</td>
<td>2.402</td>
<td>1.085</td>
<td>1</td>
<td>1.684</td>
</tr>
<tr>
<td>1600</td>
<td>545</td>
<td>0.000</td>
<td>0.000</td>
<td>0.791</td>
<td>0.000</td>
</tr>
<tr>
<td>1700</td>
<td>610</td>
<td>5.669</td>
<td>2.561</td>
<td>1</td>
<td>2.295</td>
</tr>
<tr>
<td>1800</td>
<td>900</td>
<td>3.706</td>
<td>1.674</td>
<td>1</td>
<td>1.458</td>
</tr>
<tr>
<td>1900</td>
<td>1625</td>
<td>30.565</td>
<td>13.806</td>
<td>1</td>
<td>6.759</td>
</tr>
<tr>
<td>2000</td>
<td>5333</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
</tbody>
</table>

Note: For comparability, the idea shocks in the indicated columns are normalized so that their means are equal to one. The raw shocks have means given by 2.214 for the baseline case and 0.00311 for $\phi = 3/4$. 
Growth Over the Very Long Run

where $\tilde{\rho}_t \equiv \tilde{\delta}_t / L_t^{1-\beta}$. Therefore, $\tilde{\rho}_t$ is an alternative measure of the productivity of the population at producing ideas, corresponding to the amount by which output per worker is raised through the creation of the new ideas in period $t$. Tables 8 and 9 report the decompositions in equations (23) and (24), averaged over particular time intervals.

Several remarks on the idea shocks are in order. Except where noted, these remarks will consider ideas measured in output units, as in Table 9. First, notice that on average, the productivity of the world population at generating ideas has risen substantially over time. In the year 25000 B.C., idea productivity was such that it would have required 1127 years for 1 million individuals to produce a single idea. However, by 1900, this time had fallen to 7.7 years. A useful summary of the evidence is provided by the last column in the two tables, which reports the number of new ideas generated in a single year, given the population and idea productivity in that year. Between 25000 B.C. and 1900, the number of new ideas generated by the world each year rises from 0.0022 to 33, a factor of 15000. This enormous increase in idea production is the product of two changes. First, the world in 1900 had a larger population available to produce ideas. With $\lambda = 3/4$, the effective input to research was over 100 times larger. Second, people in 1900 were more productive at generating ideas, by a factor of 150.

In fact, comparing these results to those in Table 8, one sees that the population is actually more than 1100 times more productive at producing ideas (in TFP units) in 1900 than in 25000 B.C. However, $L_t^{1-\beta}$ is also higher by a factor of 8, accounting for the difference between the rise in $\tilde{\delta}$ and the rise in $\tilde{\rho}$.

The general rise in idea productivity occurs against a backdrop of fluctuations. Between the years 5000 B.C. and 1 A.D., the world population was especially productive in generating ideas. Average idea productivity in

---

10 This is a place where the sensitivity of the quantitative results to the exact value of $\lambda$ is apparent. For example, if $\lambda = 1$, the research input rises by the same factor as population, $1625/3.34=487$. 


Table 8: Average Idea Productivity $\bar{\delta}$

<table>
<thead>
<tr>
<th>Interval</th>
<th>Average $\delta$</th>
<th>Years Needed to Generate One Idea</th>
<th>New Ideas $N_i^\lambda$ Per Year</th>
</tr>
</thead>
<tbody>
<tr>
<td>-25000 to -10000</td>
<td>0.026</td>
<td>955</td>
<td>2.47 0.0026</td>
</tr>
<tr>
<td>-10000 to -5000</td>
<td>0.093</td>
<td>268</td>
<td>2.83 0.0106</td>
</tr>
<tr>
<td>-5000 to 0</td>
<td>0.887</td>
<td>28</td>
<td>3.34 0.1187</td>
</tr>
<tr>
<td>0 to 1000</td>
<td>0.284</td>
<td>88</td>
<td>47.08 0.5353</td>
</tr>
<tr>
<td>1000 to 1500</td>
<td>0.773</td>
<td>32</td>
<td>65.68 2.0300</td>
</tr>
<tr>
<td>1500 to 1900</td>
<td>2.944</td>
<td>8</td>
<td>93.60 11.0233</td>
</tr>
<tr>
<td>1900 to 2000</td>
<td>30.565</td>
<td>0.8</td>
<td>255.94 312.9086</td>
</tr>
</tbody>
</table>

Note: Average $\delta$ is computed using the data in Table 7. “Years Needed to Generate One Idea” is the inverse of $\delta$ multiplied by the period length of 25 years. $N_i^\lambda$ is calculated using the population at the start of the interval. “New Ideas per Year” is the product of “Average $\delta$” and “$N_i^\lambda$”, divided by the period length of 25 years.

Table 9: Average Idea Productivity in Output Units: $\bar{\rho}$

<table>
<thead>
<tr>
<th>Interval</th>
<th>Average $\bar{\rho}$</th>
<th>Years to Produce Ideas s.t. $\Delta y = 1$</th>
<th>From Ideas $\Delta y$ From Ideas $N_i^\lambda$ Per Year</th>
</tr>
</thead>
<tbody>
<tr>
<td>-25000 to -10000</td>
<td>0.022</td>
<td>1127</td>
<td>2.47 0.0022</td>
</tr>
<tr>
<td>-10000 to -5000</td>
<td>0.075</td>
<td>335</td>
<td>2.83 0.0084</td>
</tr>
<tr>
<td>-5000 to 0</td>
<td>0.498</td>
<td>50</td>
<td>3.34 0.0666</td>
</tr>
<tr>
<td>0 to 1000</td>
<td>0.060</td>
<td>414</td>
<td>47.08 0.1137</td>
</tr>
<tr>
<td>1000 to 1500</td>
<td>0.152</td>
<td>164</td>
<td>65.68 0.3999</td>
</tr>
<tr>
<td>1500 to 1900</td>
<td>0.431</td>
<td>58</td>
<td>93.60 1.6152</td>
</tr>
<tr>
<td>1900 to 2000</td>
<td>3.234</td>
<td>7.7</td>
<td>255.94 33.1059</td>
</tr>
</tbody>
</table>

Note: Average $\bar{\rho}$ is computed from equation (24). “Years to Produce Ideas s.t. $\Delta y = 1$” is the inverse of $\bar{\rho}$ multiplied by the period length of 25 years. $N_i^\lambda$ is calculated using the population at the start of the interval. “$\Delta y$ from Ideas per Year” is the product of “Average $\bar{\rho}$” and “$N_i^\lambda$”, divided by the period length of 25 years.
this interval, measured by \( \bar{p} \), was .498, substantially exceeding productivity over the next 1500 years. Historically, this period marked the emergence of civilization in the form of cities. Key technological developments included writing, the beginning of scientific observation, the widespread use of metals, and dramatic improvements in transportation capabilities through the construction of ships and wagons. This pattern fits with a view that the world during the ascendency of the Mesopotamian, Egyptian, and Greek civilizations was more productive at generating ideas than during the middle and dark ages prior to the Enlightenment. The larger population of the later period compensates for this difference in productivity to some extent, but the simulation results reported later in Table 10 suggest that more new ideas were created between 5000 B.C. and 1 A.D. than between 1 A.D. and the year 1500.

Finally, the idea shock required during the 20th century to match the population data is truly remarkable; the value of 3.23 suggests that a population of a million people could produce a new idea every 7.7 years. This level of idea productivity vastly exceeds the level at any other point in history. Twentieth century productivity is more than seven times its value over the preceding four hundred years, more than 25 times its value between year 0 and year 1500, and nearly 150 times its value prior to the year 10,000 B.C. Measured in TFP units, of course, the improvement is even more substantial.

What was the shock that occurred in the 20th century? In part, the increase in productivity may reflect a change in the technological “landscape”: for example, once one discovers the periodic table and electricity, perhaps a large number of ideas become increasingly obvious and possible. In addition, however, the shock likely reflects endogenous but unmodeled features of the economy. The development of intellectual property rights over the last several hundred years surely has made a given population more productive as a larger fraction of the population, lured by the profits available to entrepreneurs, focuses effort explicitly on creating new ideas. The
evolution of the bidirectional flow of ideas between science and industry, the creation of corporate research labs, and the establishment of universities emphasizing both basic and applied research probably also underlie the large, recent idea shocks; these changes are discussed in detail by Rosenberg (1982) and Rosenberg and Nelson (1994). This kind of analysis suggests one way in which the model is testable: the resulting shocks can be compared to historical evidence.

The “negative” TFP shocks may also be analyzed in this fashion. These shocks are shown in the fifth column of Table 7 and play the following role. The basic model contains forces that, at least until the demographic transition occurs, imply an ever-increasing rate of population growth. In the data, in contrast, there are a number of periods during which population growth falls or even becomes negative. To account for these periods, we reduce TFP, which in turn reduces fertility and raises mortality. For example, between the years 200 and 600, world population increased only from 190 million to 200 million. Relative to previous rates of increase, this change was surprisingly small, and the model requires a TFP shock in which the idea stock ceased to rise and productivity in producing goods was only 94 to 98 percent of its full value. Historically, one may associate these shocks with the collapse of the Han Empire in China in 220 A.D. and the decline and fall of the Roman Empire.

A second series of TFP shocks between the years 1100 and 1400 are also required. During this interval, world population increased by only 30 million people, from 320 million in 1100 to 350 million in 1400. For comparison, world population increased by 50 million people in the three hundred years following the year 500 B.C., beginning at a much lower base of only 100 million. Between 1100 and 1400, the model requires the economy to run at only 3/4 of its full productivity potential. Significant shocks during this period include the genocidal Mongol invasions under Genghis Kahn and his successors (in which perhaps a third of the population of China died) and
the Black Death in Europe of the mid-14th century, which killed between a quarter and a third of the European population.

Finally, the model requires one more TFP shock during the 17th century. According to McEvedy and Jones (1978), population remained constant between 1600 and 1650 at 545 million people and then rose to 610 million by 1700. To match the small rate of increase over this century, the model requires a surprisingly large shock in which productivity is only 79 percent for a century. Notable shocks in this period include the Thirty Years’ War in Europe and the Manchu conquest in China.

5.1 The Shape of the Idea Production Function

The evidence on $\bar{\delta}$ suggests that on average the productivity of the population at producing ideas is rising over time. As discussed above, some of this rise, particularly in the last two hundred years, may reflect changes in property rights and the formalization of research. On the other hand, some of it may also reflect the general structure of the idea production function.

In principle, there is no reason why idea productivity should exhibit any regularity. For example, it is possible for productivity to be an increasing function of the stock of knowledge over some range or a decreasing function of the stock of knowledge or some combination of the two. Perhaps discovering the transistor leads to the semiconductor which leads to the internet. And perhaps there is nothing after that: idea productivity could be rising over some range and then fall to zero forever if there is a fixed stock of ideas that exists to be discovered.

I mention this potential lack of regularity as a warning, since we will impose regularity in what follows. Although the past may be no guide to the future, it is the only guide that we have. For many purposes, such as judging the ability of the basic model to explain the patterns in population and consumption, it is not important for us to decompose changes in idea productivity into true shocks and the portion attributable to knowledge
spillovers, e.g. associated with $A^\phi$. However, for simulating the model in the absence of shocks or for asking what the model implies about future growth such a decomposition is needed.

Figure 7 displays the technology shocks together with two curves of the form $\delta A_t^\phi$, one for $\phi = 3/4$ and the other for $\phi = 3$. The curve with $\phi = 3/4$ fits the gradual rise in idea productivity but underpredicts by an order of magnitude the sharp increase in the 20th century. The curve with $\phi = 3$ does a better job of matching the sudden rise in idea productivity in the last 200 years, but at the cost of underpredicting idea productivity over most of history.

Instead of imposing $\lambda = 3/4$ and guessing the value of $\phi$, one can estimate both $\lambda$ and $\phi$ using nonlinear least squares to minimize the sum of squared deviations of idea shocks needed to fit the population data. The result,
however, is a value of \( \lambda \) that is substantially \textit{less} than zero and a value of \( \phi \) that is substantially \textit{more} than one. For example, taking the idea shocks in Figure 7 as the data to be fit, the estimates are \( \hat{\lambda} = -0.241 \) and \( \hat{\phi} = 3.392 \).\footnote{These estimates are obtained as follows. The model to be fit is \( \delta_t = \delta N_t^\xi A_t^\phi (1 + u_t) \). Nonlinear least squares to minimize \( u' u \) produces \( \xi = -0.991 \) and \( \phi = 3.392 \). The value of \( \lambda \) reported above is \( \xi + 3/4 \).} Nonlinear least squares wants to choose a high value of \( \phi \) to match the extremely sharp rise in idea productivity in the last 200 years. The negative value of \( \lambda \) then allows the procedure to fit the average value of productivity leading up to this rise.

Ignoring the implausible estimate of \( \lambda \) for a moment, the result that one needs such a high value of \( \phi \) is somewhat surprising. Much of the growth literature has focused on the case of \( \phi = 1 \). More recently, a number of papers have emphasized \( \phi < 1 \). Interestingly, however, Romer’s original (1986) paper contains a model in which \( \phi > 1 \). Romer was explicitly motivated by the rise in growth rates in recent centuries rather than by the seemingly balanced nature of 20th century U.S. growth. To match this increase, he assumed the equivalent of \( \phi > 1 \), which in his model showed up as an increasing marginal product of knowledge. The quantitative analysis here suggests an alternative interpretation, given the implausibility of so large a value of \( \phi \) combined with a negative value of \( \lambda \).

This alternative interpretation is related to a fundamental identification problem that is not adequately resolved in simple estimation. Some of the recent rise in idea productivity almost certainly does not reflect a regular structure in the idea production function. Rather, changes in property rights, the creation of universities, and the solidification of links between science and industry have increased the productivity of the population at producing ideas. These changes might be better thought of as an increase in idea productivity given \( N \) and \( A \) than as an increase driven by the knowledge spillovers associated with \( A^\phi \). For example, the fraction of the population actively engaged in the search for new ideas surely increased during this
period, which here would show up as a change in $\delta$. In this sense, the parameter $\phi$ should not be made to fit the entire amount of the sharp rise in productivity. But unless we know exactly how much should be attributed to $\phi$ and how much should be attributed to these other factors, we will not be able to identify $\phi$ from this data.

Given this lack of identification, we will proceed by simply assuming a value. Evidence from the 20th century suggests that $\phi \geq 1$ does not fit the data: it implies that per capita growth rates should be growing with the increase in the level of research effort. Also the estimated idea shocks strongly suggest that $\phi$ is greater than zero. We pick $\phi = 3/4$ for the remaining simulations, noting that the particular value is relevant only for the out-of-sample behavior of population and consumption. The idea and mortality shocks for this specification are reported in the final columns of Table 7. Notice that the idea shocks are now closer to stationary in appearance and the TFP shocks are basically unchanged.

In the last two columns of Table 7, one sees that for $\phi = 3/4$ the productivity of the population at producing ideas rose by a factor of about 6.76/2 in the 20th century relative to the two previous centuries. One interpretation of this increase is that the institutional changes induced an increase in research intensity by the population: more people began to search actively for new ideas. Assuming the increase in $\delta$ reflects only a rise in research intensity, the extent of this rise can be inferred from $\delta_{t}^{1/\lambda}$. Then, $(6.76/2)^{4/3} \approx 5$ produces the reasonable result that the fraction of the population actively engaged in the search for new ideas increased by a factor of about five in recent centuries.

5.2 Simulation Results

Figure 8 displays the actual and simulated data for population growth and consumption growth for the case of $\phi = 3/4$. Up until the year 2000, the model is simulated to fit the level of population (and therefore its growth
That the simulation fits the population data exactly is in one sense not surprising — the shocks were chosen exactly for this purpose. What is remarkable, however, is that this fit is achieved with shocks that appear reasonable given the historical record. For example, imagine the shocks that would be required for a standard neoclassical growth model to fit these same facts.

Two additional features of Figure 8 are worth noting. First, the time path of consumption growth broadly matches that outlined in our “facts” section: per capita consumption growth is quite close to zero until recent years, at which point it spikes up to nearly three percent per year. Second, regarding the future of population growth and consumption growth, both peak sometime shortly after the year 2000 and then decline, eventually to
Table 10: Simulation Results

<table>
<thead>
<tr>
<th>Year</th>
<th>A</th>
<th>N</th>
<th>Growth</th>
<th>c</th>
<th>Growth</th>
</tr>
</thead>
<tbody>
<tr>
<td>-25000</td>
<td>648</td>
<td>3</td>
<td>...</td>
<td>270</td>
<td>...</td>
</tr>
<tr>
<td>-10000</td>
<td>690</td>
<td>4</td>
<td>0.00001</td>
<td>271</td>
<td>0.00000</td>
</tr>
<tr>
<td>-5000</td>
<td>747</td>
<td>5</td>
<td>0.00004</td>
<td>272</td>
<td>0.00000</td>
</tr>
<tr>
<td>-500</td>
<td>2687</td>
<td>100</td>
<td>0.00067</td>
<td>352</td>
<td>0.00006</td>
</tr>
<tr>
<td>0</td>
<td>2687</td>
<td>170</td>
<td>0.00106</td>
<td>298</td>
<td>-0.00333</td>
</tr>
<tr>
<td>1000</td>
<td>3370</td>
<td>265</td>
<td>0.00044</td>
<td>328</td>
<td>0.00010</td>
</tr>
<tr>
<td>1500</td>
<td>4298</td>
<td>425</td>
<td>0.00094</td>
<td>358</td>
<td>0.00018</td>
</tr>
<tr>
<td>1600</td>
<td>5491</td>
<td>545</td>
<td>0.00249</td>
<td>322</td>
<td>-0.00106</td>
</tr>
<tr>
<td>1700</td>
<td>5491</td>
<td>610</td>
<td>0.00113</td>
<td>408</td>
<td>0.00236</td>
</tr>
<tr>
<td>1800</td>
<td>8280</td>
<td>900</td>
<td>0.00389</td>
<td>550</td>
<td>0.00298</td>
</tr>
<tr>
<td>1900</td>
<td>11659</td>
<td>1625</td>
<td>0.00591</td>
<td>643</td>
<td>0.00157</td>
</tr>
<tr>
<td>2000</td>
<td>71646</td>
<td>5333</td>
<td>0.01188</td>
<td>3101</td>
<td>0.01574</td>
</tr>
<tr>
<td>2100</td>
<td>1224120</td>
<td>23114</td>
<td>0.01467</td>
<td>42450</td>
<td>0.02617</td>
</tr>
<tr>
<td>2200</td>
<td>16403012</td>
<td>22663</td>
<td>-0.00020</td>
<td>579630</td>
<td>0.02614</td>
</tr>
<tr>
<td>2300</td>
<td>87058382</td>
<td>22524</td>
<td>-0.00006</td>
<td>3082678</td>
<td>0.01671</td>
</tr>
</tbody>
</table>

Note: Simulation results assuming $\phi = 3/4$. 

Growth Over the Very Long Run
zero since we have assumed that $\tilde{b} = \tilde{d} = 0$.

The simulation results in Table 10 show that the level of per capita consumption rises slightly from 25000 B.C. until the year 0. In contrast, the level of population rises from 3.34 million to 170 million, a 50-fold increase. This outcome can be interpreted with the help of the production technology. Recall that in the absence of shocks per capita consumption is given by $c_t = w_t \ell_t = \ell_t^\beta A_t^\sigma / N_t^{1-\beta}$. Therefore, if $c_t$ and $\ell_t$ are roughly constant, then $N_t$ is proportional to $A_t^{\sigma/(1-\beta)}$. That is, the population is proportional to total factor productivity $A$ raised to a power given by the inverse of the degree of diminishing returns associated with the presence of land. Here, $N$ is proportional to the cube of total factor productivity. According to the model, the stock of ideas rises by a factor of $2687/648 = 4.15$ during the 25000 year period, suggesting an increase in population by a factor of $4.15^3 = 71.5$; this is slightly more than the 50-fold increase observed, reflecting the slight increase in per capita consumption. Relatively small changes in the stock of ideas have the power to generate large changes in population in the model.

The levels of population and consumption are shown in Figure 9. The apparent constancy of consumption for most of history suggested in the figure is an artifact of the time scale. Figure 10 plots the level of per capita consumption from 1000 B.C. until 1800 A.D. to illustrate this point. From an average level of $270$ throughout most of time, per capita consumption rises to $300$ in 1000 B.C. and reaches a local peak of about $352$ in 500 B.C. before falling back to $298$ by year 0. Reasonably large swings in consumption similar to this one continue through the year 1800, reflecting the impact of idea and TFP shocks. Some of these swings also seem to be related to historical events. Drawing on the work of Thorold Rogers, Clark (1940, p. 168) suggests that standards of living stabilized at a local maximum in the 15th century and then declined through the 17th century, a feature replicated by the simulation.
Returning to the broader pattern of population and consumption displayed in Figure 9, one sees a rapid rise in consumption around the year
2000, leading to the onset of the fertility transition. World population stabilizes at 23 billion around the year 2100.$^{12}$

These patterns of population and consumption growth can be seen more clearly in Figure 11, which focuses in on a 600 year period beginning with the 20th century. Population growth peaks in the year 2025, coming much closer to the actual peak in world population growth that seems to have occurred during the 1960s. With $\phi = 3/4$, the measure of increasing returns to scale in this economy is large ($\theta = 2.67$) so that consumption growth peaks at a value substantially higher than the peak population growth rate. Recall that world GDP growth seemed to peak in the 1960s at around 3 percent per year. While the timing of the peak is off (by more than a century), the magnitude is about right for $\phi = 3/4$.

$^{12}$These values are quite sensitive to the value of $\phi$. For example, with $\phi = 0$, consumption rises more gradually, delaying the onset of the decline in fertility. As a result, world population grows (implausibly) to more than 400 billion before stabilizing!
Two discrepancies between the data and the simulation deserve mention. First, the peak in both population growth and consumption growth occurs substantially later in the simulation than in the data. Second, in the data, the peak in population growth and the peak in consumption growth seem to occur at about the same time (of course it is difficult to know this without observing additional data; the 1960s could be a shock rather than the true peak).

6 Was the Industrial Revolution Inevitable?

A sensible working definition of an industrial revolution for this model is a substantial and rapid rise in both the level and growth rate of per capita consumption accompanied by a rise in population growth and followed by a demographic transition. Based on this definition, the model suggests that an industrial revolution was indeed inevitable, at least for the parameter values under consideration. This was apparent in Figure 5.

But was the Industrial Revolution inevitable? If by the Industrial Revolution we mean the onset of rapid population and per capita growth culminating in the large increases in standards of living during the 20th century, then the answer turns out to be no.

To see this, consider the following counterfactual experiment. Suppose the idea shocks from the 18th century onward are shut off. Specifically, the value of $\delta_t$ is set equal to its mean starting in the year 1700. The simulation results for this case are reported in Figure 12. What we see from this experiment is that an industrial revolution does indeed occur, but it is delayed by more than 400 years for the case of $\phi = 3/4$.

So was the Industrial Revolution (with capital letters) inevitable? Not if one means the increases in standards of living that occurred from 1760 to the present. What the model wants to call idea shocks, and what the economic historian might call something else, played a critical role in the timing of
the Industrial Revolution that the world experienced. At the same time, it is useful to note that there were forces at work in the model, and perhaps in the world, that suggest that something like an industrial revolution may have only been just a matter of time.

7 Conclusion

This paper provides a model of growth over the very long run in which the basic story goes something like the following. A long time ago, the world population was relatively small and the productivity of this population at producing ideas was relatively low. For example, in the year 25000 B.C., the model suggests that it took 300 years before the society of 3.34 million people produced a single new idea. Once this idea was discovered however, consumption and fertility rose, producing a rise in population growth, so that there were more people available to find new ideas, and the next new idea was discovered more quickly. In the model, this feedback leads to accelerating rates of population growth and consumption growth provided the aggregate production technology is characterized by increasing returns to accumulable factors.
In the absence of shocks, this general feedback seems capable of producing something like an industrial revolution. However, the quantitative analysis suggests that shocks which raised the productivity with which the world population produces ideas have been extremely important in generating the observed Industrial Revolution. Rather than operating through a stable knowledge spillover, it seems plausible that an important effect of these shocks was to increase the intensity with which the population searches for ideas: the establishment of property rights, for example, made searching for new ideas an increasingly lucrative activity.

The resulting technological progress and the rise in consumption led to a reduction in mortality followed by a reduction in fertility as the demographic transition sets in. In the very long-run, it is possible for the level of the population to stabilize while the level of consumption grows to infinity, albeit at a growth rate that gradually falls to zero.

8 Appendix

8.1 Existence and Uniqueness of the Static Equilibrium

Proof:

The first order condition in equation (11) can be combined with the two constraints in (3) and (4) to yield an implicit labor supply function $\ell(w)$:

$$
(\alpha(1 - \ell_t) - \bar{b})^\eta = \frac{\alpha \mu}{1 - \mu} \frac{(w_t \ell_t - \bar{c})^\gamma}{w_t}.
$$

(25)

The wage is determined by the production function for consumption goods. Rewriting equation (6) with $L_t = \ell_t N_t$ yields

$$
w_t = \frac{A_t^\eta \epsilon_t}{N_t^{1-\beta} \ell_t^{1-\beta}}.
$$

(26)

Equations (25) and (26) can be combined to get a single nonlinear equation that characterizes the equilibrium value of $\ell$. Define

$$
F(\ell) \equiv (\alpha(1 - \ell) - \bar{b})^\eta - \frac{\alpha \mu}{1 - \mu} \frac{1}{a} (a\ell^\beta - \bar{c})^\gamma \ell^{1-\beta}.
$$
Then the equilibrium satisfies $F(\ell^*) = 0$.

To see that there is a unique solution to this equation, first note that the Inada-type conditions on the utility function guarantee that a solution, if it exists, must satisfy $\bar{c} > 0$ and $\bar{b} > 0$. In terms of $\ell$, these conditions imply that $\ell > (\bar{c}/a)^{1/\beta}$ and $\ell < 1 - \bar{b}/\alpha$. Therefore, we require $(\bar{c}/a)^{1/\beta} < 1 - \bar{b}/\alpha$ in order for a solution to exist. Given the definition of $\bar{a}$, this puts restrictions on initial conditions.

Next, notice that $F(\bar{c}/a) > 0$ and $F(1 - \bar{b}/\alpha) < 0$. Therefore, provided $F(\ell)$ is monotonicly decreasing within this range, the solution is unique. The condition that $F'(\ell) < 0$ for $(\bar{c}/a)^{1/\beta} < \ell < 1 - \bar{b}/\alpha$ is readily verified. Once $\ell^*(a)$ is determined, the remaining quantities in the proposition are given in a straightforward fashion from equations (26), (3), and (4). Q.E.D.

### 8.2 Solving for Productivity Shocks

Given the parameter values in Tables 2, 4, and 6, and given the population data in Table 1, we solve for the sequence of idea shocks $\{\delta_t\}$ and TFP shocks $\{\epsilon_t\}$ where $t = 0$ corresponds to the year 25000 B.C., and each unit increment to $t$ corresponds to an increment of 25 years. The solution is obtained as follows:

1. We begin with an initial population, an initial stock of ideas, and an observation for population some periods later. Let $NumPeriods$ denote the number of periods between the two observations on population. For example, if the first period corresponds to the observation in the year -2000 and the next is the year -1000, we have $1000/25 + 1 = 41$ periods.

2. Solve for the constant value of the shock $\tilde{\delta}$ such that the dynamics of the model would lead population to grow from its level at the first observation to its level at the second observation after $NumPeriod$ periods, with a percentage error less than or equal to $10^{-8}$. If such
a value is found and is “small” in the sense that it does not involve passing through the entire demographic transition in one period, then we’re done with this step. Set the mortality rate shock for period $NumPeriod$ equal to zero since it is not needed.

3. With respect to the previous step, there are two things to note. First, there are occasionally multiple values of the $\tilde{\delta}$ shock that will work. We choose the smallest value (so that we are on the pre-demographic transition side of the population growth schedule as much as possible). Second, for declines in the level of population, or for relatively small increases, it is possible that no “small” shock will work. In this case, set the idea shock for the periods corresponding to $1 : (NumPeriods - 1)$ equal to zero, and solve for the reduction in TFP — the constant value of $\epsilon < 1$ — such that the simulation matches the level of population after $NumPeriod$ periods, with a percentage error less than or equal to $10^{-8}$.

4. Advance to the next population observation and repeat this process, starting with step 1 above, until all population observations have been fit by the model.
References


Kortum, Samuel S., “Research, Patenting, and Technological Change,” 


